ON THE

ARITHMOMETER

OF

M. THOMAS (DE COLMAR),

AND ITS

APPLICATION TO THE CONSTRUCTION OF

LIFE CONTINGENCY TABLES.

BY

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To Chas. Henry Bannerman,

from

the Author.

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INTRODUCTION.

The main object of this little tract, (which is reprinted from the *Journal of the Institute of Actuaries*,) is to bring to the knowledge of computers, and generally of those having to do with calculations, the assistance they may derive in their work by the employment of the Arithmometer or Calculating Machine of M. Thomas (de Colmar). Abundant illustration of the applications of the machine will be found in the following pages; and it is proposed to give here a brief statement of its capabilities and mode of operation.

M. Thomas's machine is provided, on the face, with spaces for the reception of three numbers, which may be denoted by P, Q, and R, respectively. Two of them, P and Q, being set in their places on the machine, the regulator adjusted (by a touch of the finger), and the manual operation performed, R appears on the space allotted for it, and on the space appropriated to P we find now the numerical value of $P \pm QR$; that is, P increased or diminished by the product QR, according as the regulator was adjusted for addition or subtraction. The number Q remains in its place.

The manual operation, the same in all cases in character, although not in extent, consists in the turning of a handle, with some intermediate movements. The number of turns requisite in any particular case is the sum of the digits composing the number R; and it will average about four-and-a-half times the number of places in R.

The time occupied in the performance of its operations is an element of importance in forming a judgment as to the utility of the machine. In elucidation of this point, it may be mentioned that

when	P=	8,72540,93671,
	Q=	4,28954,
and	R=	1,89208,

one minute suffices for the production, (*free from error,*) of either of the following results:—

$$P+QR=16,84156,22103,$$
$$P-QR=\quad 60925,65239,$$

about half the time in each case being occupied in setting the numbers on the machine.

But it is, as will be presently shown, in its application to the construction of tables that the advantage gained, in point of accuracy and despatch, by the employment of the machine is fully realized.

It seems hardly necessary to observe that the power the machine has been shown to possess includes that of performing all the fundamental operations of arithmetic. Thus, from $P+QR$, when $R=1$ we have $P+Q$, and from $P-QR$, in the same circumstances, we get $P-Q$. Also, when $P=0$ we have from $P+QR$ the product QR. As regards division the case is somewhat different. P, the dividend, and Q, the divisor, being given, we have to find R, the quotient. The form of the operation is $P-QR$, and we multiply Q by such digits, in orderly succession, as visibly to reduce P to 0, or to a number less than Q; R, the quotient (being the aggregate of the multipliers), is then found in its place, and the remainder, if there be one, in the place of P.

Although there are, no doubt, forms of numerical tables in the construction of which the Arithmometer is not directly applicable, there are few in connection with which it may not, in proper hands, be turned to useful account. There are three forms to which it readily adapts itself.

The first form comprises those tables in which the series to be formed consists of the products or the quotients of corresponding terms of two independent series. To this class belong most of the tables connected with the social and physical sciences; such as tables of averages, specific gravities, refractive powers, &c. An example of a formation belonging to this class will be found on p. 31.

The work requisite in a formation of this class for each term formed, is, (*a*) when the required result is QR, to set on Q, multiply by R, record the result, efface (P=) QR and R; (*b*) when the result required is P÷Q or R, to set on P and Q, multiply by R, record R, efface P (if necessary) and R.

The second form of table in the construction of which the Arithmometer can be advantageously employed, is that in which the terms to be formed are so related that the difference between succeeding terms is the product of corresponding terms of two given series.

The work requisite for each term here is the same as in the last case, (*a*), with the exception that here R only needs to be effaced. The number found in the place of P always remains on the machine. The work is, consequently, less than in the former case, although the function here formed is more complex.

Several examples of this mode of formation will be found in the following pages. *See*, in particular, pp. 13, 15, 29, &c.

The third form of table, being that in which the greatest advantage of all is derived from the employment of the machine, is the same as the second, with the exception that here the factor Q is constant. There is consequently, after the first term has been formed, no setting on the machine at all, and the whole work consists in a series of multiplications by the successive values of R (in this case almost necessarily small numbers), interrupted only by the recording of the results as they arise in the place of P, and the effacing of the used values of R.

Many examples of this mode of formation are shown in the tract, as on pp. 5, 7, 9, 10, 19, 22, 25, 32, &c.

The preceding sketch would be imperfect as an enumeration of the capabilities of the Arithmometer, were mention withheld of the facility it affords for the correction of errors that may have been committed in the course of an operation. Thus if, for example, an erroneous value of R (discoverable by comparison of the value appearing on the machine with the value as given) shall be found to have been made use of, the error can be easily and promptly rectified, whether it extend to one or to many figures.

LONDON, 29*th June*, 1874.

ON THE

ARITHMOMETER,

ETC., ETC., ETC.

THE Arithmometer of M. Thomas (de Colmar) has been already brought under the notice of the readers of this *Journal* by General Hannyngton, in a remarkably lucid and suggestive paper, which will be found at p. 244, vol. xvi. General Hannyngton, in his paper, explains the manner of working the machine, and gives examples of some of its applications to the construction of actuarial tables, with hints as to others. These afford an idea of the very striking adaptation of the machine to the formation of such tables; and they cannot fail to have excited the interest of many of the readers.

The present paper is intended to be supplemental to that of General Hannyngton, and in it the adaptation in question will be further shown. An attempt will also be made to systematize the manner of its application, and detailed examples in illustration will be given. There will be no need to say anything here as to the actual working of the machine, since this has been so well explained by General Hannyngton. There are, nevertheless, certain points in the manipulation to which it may be well to advert in the outset.

It is usual to describe the Arithmometer as a machine which enables a person, however unskilled himself, to perform the operations of multiplication and division with facility, rapidity, and unfailing accuracy. This, as a description, is correct as far as it goes; but as an enumeration of the properties of the machine, it is inadequate and defective. It entirely omits that property which forms its special adaptation to our purpose, and in default of which its utility would be comparatively limited. Besides the facilitation of the operations named, the machine will also, in forming the product of two given numbers, either add that product to, or subtract it from, another given number, according to the pleasure of the operator. Abundant illustration of the application of this property will be found in the present paper.

There are three forms, then, to the numerical evaluation of which the Arithmometer is directly applicable. These may be symbolized as follows:—

$$QR, \quad \frac{Q}{R}, \quad \text{and } P \pm QR;$$

P, Q and R denoting any given numbers.

Familiar instances of these forms, in the present connexion, are:—

	QR	$D_x = l_x v^x$;
	$\frac{Q}{R}$	$a_x = \frac{N_x}{D_x}$;
and	$P + QR$	$N_x = N_{x+1} + l_{x+1} v^{x+1}$;
	$P - QR$	$A_x = 1 - (1 - v)(1 + a_x)$.

The manner of applying the machine to the evaluation of expressions of the first and second forms; in other words, the manner of performing the operations of multiplication and division, has been sufficiently explained in General Hannyngton's paper. Of its application to the third form some further elucidation is necessary.

The special adaptation of this form to the construction of tables may be shown as follows:—

If u_x be any function of x, we always have,

$$u_{x+1} = u_x + \Delta u_x;$$

and this is of the form $P \pm QR$ if Δu_x, the difference of u_x, can be exhibited as the product of two given numbers. In such cases, then, (and it will soon appear that they are by no means rare,) we have, in forming a series of successive values, the benefit of a continuous process, with its attendant advantages. The result of each operation becomes the P of the next, and we pass from a preceding to a succeeding value by adding to (or subtracting from) the former the product of two known numbers. And this, as we have seen, is a function to which the Arithmometer most readily lends itself.

In the applications of the machine there are thus, it appears, three numbers to be dealt with; and provision is accordingly made on it of three spaces for their separate exhibition. Two of these spaces are on the *slide*, and the third on the *face*. I propose to designate them by S_1, S_2, and F, respectively; and the numbers occupying the several spaces will be denoted by the same symbols, respectively distinguished, however, when so used, by being enclosed in parentheses, thus:—(S_1), (S_2), (F).

In employing the formula $P \pm QR$, the number P is placed on S_1, and Q (the *in* factor), upon F; multiplication being then made by R (the *out* factor), this number appears upon S_2, and the result, which is the required value of $P \pm QR$, replaces P upon S_1.

A point in regard to which difficulty is felt, in commencing the use of the Arithmometer, is the setting of the numbers P and Q upon the machine, so that the product QR shall, when formed, fall in its proper place with respect to P. The rule for this purpose (which has not hitherto been given) is very simple. It is:—

Draw out the slide as many holes as there are decimal places in R, the *out* factor; and place P and Q upon S_1 and F respectively, in such a manner as that like denominations shall stand under (and over) like. The slide is then to be pushed home; and, the multiplication being performed, the correct result will appear on S_1.

If $P=0$, that is, if it is only the product QR that is required, no preliminary drawing out of the slide is necessary.

So far as the results of the formulæ QR, $P \pm QR$ are concerned, it is obviously indifferent which of the two, Q, R, we employ as the *in* factor; in each case of practical application, however, there are usually circumstances sufficient to determine our choice in favour of one or the other. I shall in general, when necessary for distinction, let Q denote the *in* factor—the factor to be set upon F; and R, of course, to denote the *out* factor—the factor to be employed as a multiplier.

The points that have been now more specially adverted to, and others that may arise, will find ample illustration in the examples appended to the problems to which I now proceed.

The examples will be taken from *The Institute of Actuaries' Life Tables*, H^M mortality, and three per-cent interest.

PROBLEM I.—Given a table of annuities; to form the corresponding table of assurances.

We have, $$A_x = 1-(1-v)(1+a_x) \quad . \quad . \quad . \quad . \quad . \qquad (1)$$
$$\therefore \quad \Delta A_x = -(1-v)\Delta a_x;$$
whence, $$A_{x+1} = A_x + \Delta A_x = A_x - (1-v)\Delta a_x \quad . \quad . \quad . \quad . \qquad (2)$$
By these formulæ the required table will be constructed, (1) coming into use for the formation of the initial term, and (2) for the continuous formation of the subsequent terms.

Example 1.—The given table of annuities is that on single lives, p. 14 (*Institute Tables*); and it is required to reproduce the table of assurances in the adjoining column.

Here we have, by (1),

$$P = \quad 1{\cdot}0000000,$$
$$Q = 1-v = \quad {\cdot}0291262,$$
$$R = 1+a_{10} = 25{\cdot}1484.$$

We here take $1-v$ for Q, because it remains constant during the process; and we use six significant figures because, there being also six in R, we desire to have our results true to the same extent.

P and Q are placed on the machine as follows:—

S_1 0 0 0 0 1 0 0 0 0 0 0 0 0 | 0 0 0 0
F 0 0 2 9 1 2 6 2 |

There are *four* decimal places in R; *therefore*, by the rule, the denominations of the figures in P and Q are made to correspond when the slide is drawn out *four* holes.

One of the ivory pegs which accompany the machine is now inserted in the place of the decimal point, and another between the sixth and seventh decimal places; the last-named place also (the first beyond the peg) is increased by 5. The pegs serve to direct the eye to the part of the slide where the figures to be taken out for record are to be found; and the effect of the addition of 5 in the *next* place being to impart to the adjoining figure the usual correction, when necessary,* the eye, in taking out the result, has never to travel beyond the space embraced by the two pegs.

I repeat the representation of the setting of the numbers on the machine for the present example, adding indications of the positions of the pegs and of the increased figure:—

0 0 0 0 1,0 0 0 0 0 0,5* | 0 0 0 0
0 0 2 9 1 2 6 2 |

The same indications will be used in subsequent representations of the setting of the machine.

When the slide is pushed in as directed, the above setting will assume the following appearance:—

0 0 0 0 1,0 0 0 0 0 0,5* 0 0 0 0 |
0 0 2 9 1 2 6 2 |

The form here being P−QR, and both the factors Q, R, being positive, the regulator is set for subtraction; multiplication is then made by R=251484, and the result is,

0 0 0 0 0,2 6 7 5 2 3,1* 7 1 9 2.

* The method here made use of for the correction of the last figure in the terms of series formed by addition (or subtraction) was, I believe, first suggested and exemplified by me, in a work published in 1849. Among the instances of its advantageous employment that have come to my knowledge, the most striking is one that occurred to Dr. Farr, in the construction of his tables of log v^x (*Tables of Life-Times*, pp. 6–11). These tables were formed by the aid of Scheutz's machine; a specialty of which is, that it records its results in the form of a mould to be employed as the matrix for a stereotype cast. It is consequently necessary, in the use of this machine, that each result, *as it arises*, should be correct in the last figure to be recorded. Dr. Farr informs us, in his Introduction (p. cxliii), that the method he employed for this purpose was that now under consideration. The required correction could probably, in this case, have been given in no other way.

Had the increase of 5 in the fifth place from the last not been made, the figures which now read 3,1, would have read 2,6; and in transcribing the result we should have had to correct it in the last place. The correction made in this manner once for all, suffices for the whole of the succeeding values; and, as has been already said, we shall not have occasion to look at any figures except those in the space between the pegs.

The expression (2) now comes into use for the completion of the series. It is,

$$A_{x+1}=A_x-(1-v)\Delta a_x;$$

which when $x=10$ becomes,

$$A_{11}=A_{10}-(1-v)\Delta a_{10}.$$

This also is of the form P−QR; but here R, that is Δa_{10}, being negative, the regulator must be set for addition. In other respects the setting of the machine remains; and after effacing (S_2), the series is completed by employing as multipliers the successive terms of Δa_x, effacing each, of course, after it has been used.

The series Δa_x should be formed on a separate slip of paper or cardboard (and proved by addition), as it comes into use in other formations.

The commencing portion of the results, as they come out, is here given; and it may be compared with the corresponding portion of the series* on p. 14 of the Institute volume.

x	Δa_x	A_x
10		·267523
11	1531	·271982
12	1811	·277257
13	2039	·283195
14	2206	·289621
15	2316	·296366
16	2366	·303258
17	2354	·310114
18	2284	·316766
19	2148	·323023
20	2004	·328860
	1941	
*	*	*

* The comparison here suggested will reveal the existence of a few slight discrepancies in the last place between the two sets of values compared. Both nevertheless are correct deductions from the data employed in their formation. The discrepancies, such as they are, originate in the relation which subsists between the values of an annuity and an assurance on the same status. It can easily be shown that, *at three per-cent,* four-decimal annuities are barely sufficient for the accurate determination of six-decimal assurances.

Although the machine (when in order) does not commit mistakes, the operator enjoys no such immunity from error. He may, for instance, have used a wrong figure in one of the multipliers;* and the error thus committed will, unless discovered and corrected, vitiate the succeeding values. It is, therefore, necessary to be provided with means for its speedy detection. These will be three or four terms of the series, at equal intervals, which may be readily formed in the same manner as the initial term. Each of these, as it is reached, will form a check on all the preceding work.

The series A_x may be formed with the same facility if we commence the formation with the oldest instead of the youngest tabular age. The chief difference in the working will be, that as in this order Δa_x is positive, the regulator will remain at subtraction throughout. The initial term will be

$$\begin{aligned} A_{97} &= 1-(1-v)(1+a_{97}) \\ &= 1-(1-v)\times 1{\cdot}0000, \end{aligned}$$

since $a_{97}=0$; and it will be formed, after setting on P and Q as for A_{10}, by a single turn, *without pushing in the slide.*

Example 2.—Let the given annuity series be that in column 64, p. 150; to form the corresponding assurance series.

The process here is in such entire analogy to that exemplified in Example 1, that it will be sufficient, after two remarks, to give a specimen.

The first remark is, that while, as a rule, the differences of the annuities are negative, there are exceptions; as here $\Delta a_{64 \cdot 10}$ is positive. Differences thus abnormally affected should be written in red ink; and this will serve as a warning, when using them, to alter the regulator from its normal position.

The second remark is, that, whereas the differences of the single-life annuities consist generally of four significant figures, those of the joint-life annuities, as here arranged, have never more than three. The assurances corresponding to these will therefore admit of construction at the cost of a proportionally less amount of labour. I find, in fact, that a column (45 values), after the differences and verifications have been formed, need not occupy in construction (including the recording), more than twenty minutes. It would hence be quite a practicable task to form the assurances

* The facility with which errors of this kind, *when noticed at the time,* admit of correction, forms one of the great merits of the Arithmometer. The erroneous figure being brought into the working position, it is set right by the requisite number of turns; remembering that, preparatory thereto, if it is diminution that is required, the regulator must be reversed.

corresponding to the joint-life and last-survivor annuities in the Institute volume.

The following is the specimen above referred to :—

x	$\Delta a_{64.x}$	
10		·723150
11	32	·723056
12	58	·723225
13	136	·723621
14	196	·724192
15	239	·724888
16	263	·725654
17	264	·726423
18	238	·727117
19	186	·727658
20	121	·728011
	87	
*	*	*

PROBLEM II.—A table of annuities being given, it is required to construct a table of the Values of Policies at all ages, and for all durations.

The technical "Value of a Policy" is the difference between the value, on an anniversary of its inception (and consequently when a premium is just due), of the sum assured (supposed a unit) and the premium payable in respect of it, on the supposition that this is the *net premium* according to the table used in the valuation.

Denoting, for the present purpose, the value of a policy on a life now aged x, which was effected at age w, by $V_{w]x}$*, it is known that,

$$V_{w]x}=1-\frac{1+a_x}{1+a_w}$$

This may be written,

$$V_{w]x}=1-(1+a_w)^{-1}.(1+a_x)\,;$$

so that by using the reciprocal of $1+a_w$ the indicated division is changed into a multiplication. The expression is now of the form $P-QR$; and it is consequently fitted for the application of the Arithmometer.

The given annuity table being, as before, that on p. 14, the following scheme will facilitate the comprehension of the order of construction :—

* I find that the symbol for the value of a policy, ${}_nV_x$, given in the recognized notation, is unsuited when, as in the present investigation, it is necessary to treat the value as a function of the ages at entry and at valuation. I have therefore been obliged to devise a new symbol; and the one in the text, while answering my present purpose sufficiently well, is so distinctive that there is no risk of its being confused with any other symbol.

x	10	11	12	13	14	15	16
10	10·10						
11	11·10	11·11					
12	12·10	12·11	12·12				
13	13·10	13·11	13·12	13·13			
14	14·10	14·11	14·12	14·13	14·14		
15	15·10	15·11	15·12	15·13	15·14	15·15	
16	16·10	16·11	16·12	16·13	16·14	16·15	16·16

Here the values of x, the present age, are at the side, and those of w, the age at entry, at the top; so that w is constant in the columns, and x in the rows.

We have, $$V_{w]x}=1-(1+a_w)^{-1}.(1+a_x) \quad . \quad . \quad . \quad . \qquad (3)$$

It is most convenient to effect the construction in columns. Hence, making x the variable,

$$\Delta_x V_{w]x}=-(1+a_w)^{-1}.\Delta a_x;$$

and, $$V_{w]x+1}=V_{w]x}-(1+a_w)^{-1}.\Delta a_x \quad . \quad . \quad . \qquad (4)$$

The similarity of these expressions to those that arose in Problem I. is apparent; the only difference being that in these $(1+a_w)^{-1}$ takes the place of $1-v$ in the others. The process of construction, accordingly, in each column, after the formation of the initial term, is absolutely identical with that of Problem I.

The initial term for the first column is

$$V_{10]10}=1-(1+a_{10})^{-1}.(1+a_{10});$$

so that we have,

$$P= \qquad 1\cdot0000000,$$
$$Q=(1+a_{10})^{-1}= \cdot0397640,$$
$$R=1+a_{10} \quad =25\cdot1484;$$

and the setting of the machine will be as follows:—

```
               *
0 0 0 0 1,0 0 0 0 0,5 0|0 0 0 0
        0 0 3 9 7 6 4 0|
```

From (3) it appears that when $x=w$, $V_{w]x}=0$. That is, the initial term in all the columns is 0; and we might therefore assume this value, and commence the construction in each column with the formula (4). It is better, however, to proceed regularly, as we should not otherwise have that perfect identity which ought to subsist, between the values successively formed and those formed for verification in the manner to be presently shown. The reciprocals are only approximately, not absolutely true; and $V_{w]x}$ when $x=w$

never comes out exactly equal to 0, although, using the reciprocals to six significant figures, the deviation will in no case affect the *sixth* place of our results by more than a unit.

The numbers being placed on the machine as above, pushing in the slide, setting the regulator for subtraction, and multiplying by 251484, we get for $V_{10]10}$,

$$0\,0\,0\,0\,0,0\,0\,0\,0\,0,\overset{*}{4}\,0\,2\,2\,4\,0.$$

The regulator is now changed to addition, Δa_x being negative throughout, and multiplication being made by the terms of this series in succession, commencing with $\Delta a_{10}=1531$, column 10 is completed.

The other columns are formed in the same way, the initial values of w and x in each succeeding column being respectively greater by unity than in the column preceding.

The following is a specimen of the formation, embracing the commencement of the first six columns.

		397640	400075	402995	406334	410009	413940
w		10	11	12	13	14	15
10	Δa_x	·00000					
11	1531	·00609	·00000				
12	1811	·01329	·00725	·00000			
13	2039	·02140	·01540	·00822	·00000		
14	2206	·03017	·02423	·01711	·00896	·00000	
15	2316	·03938	·03349	·02644	·01837	·00950	·00000
16	2366	·04879	·04296	·03598	·02799	·01920	·00979
17	2354	·05815	·05238	·04546	·03755	·02885	·01954
18	2284	·06723	·06152	·05467	·04683	·03821	·02899
19	2148	·07577	·07011	·06332	·05556	·04702	·03788
20	2004	·08374	·07813	·07140	·06370	·05524	·04618
	1941						
*	*	*	*	*	*	*	*

The reciprocals* that come into use in the formation of the several columns, being those of $1+a_{10}$, $1+a_{11}$, &c., are here, for illustration, written at the top of the respective columns; and the slip containing the series Δa_x is represented by the side of column 10. It may, as each column is completed, be moved forward to the next.

And now, as to the formation of verifications. Resuming the expression (3)

$$V_{w]x}=1-(1+a_w)^{-1}.(1+a_x)$$

if in this we make w the variable, the values indicated will be those

* The reciprocals for the entire annuity column had better be formed at the outset, either by the machine or by Oakes's Table.

occupying the row opposite x. We may therefore, by choosing three or four values of x at suitable intervals, and forming the corresponding horizontal series, thus obtain the requisite values for verification of the work in the columns. Taking the difference with respect to w, therefore, we have,

$$\Delta_w V_{w]x} = -(1+a_x).\Delta(1+a_w)^{-1};$$

whence, $$V_{w+1]x} = V_{w]x} - (1+a_x).\Delta(1+a_w)^{-1} \quad . \quad . \quad . \quad (5)$$

Making $x=20$ and $w=10$, we have for the initial term of the row opposite 20,

$$V_{10]20} = 1-(1+a_{20})(1+a_{10})^{-1},$$

and for the next term

$$V_{11]20} = V_{10]20} - (1+a_{20}).\Delta(1+a_{10})^{-1};$$

the series being continued by using as multipliers the successive terms of the series $\Delta(1+a_w)^{-1}$, which is deduced from the values placed at the tops of the several columns in the last formation.

For the initial term:—

$$P= \qquad 1{\cdot}0000,$$
$$Q=1+a_{20} \quad =23{\cdot}0425,$$
$$R=(1+a_{10})^{-1}= \quad {\cdot}0397640;$$

and P and Q are placed on the machine as follows:—

```
                   *
0 0 0 0 1,0 0 0 0|0,5 0 0 0 0 0
  0 0 2 3 0 4 2 5|
```

There being *seven* decimal places in R, the denominations in P and Q are made to correspond when the slide is drawn out *seven* holes.

Pushing in the slide, setting the regulator for subtraction, and multiplying successively by $(1+a_{10})^{-1}=397640$, and the differences of this series, the terms in line with $x=20$ come out as follows:—

w	$\Delta(1+a_w)^{-1}$	$V_{w]20}$
10		·08374
11	2435	·07813
12	2920	·07140
13	3339	·06370
14	3675	·05524
15	3931	·04618
16	4094	·03675
17	4156	·02717
18	4109	·01770
19	3940	·00862
20	3742	·00000
	3686	
*	*	*

The whole of the work in Columns 10 to 20, down to $x=20$, is thus verified. And in the same way verification may be obtained at as many points as we please, by forming the requisite horizontal series. These ought to be formed first, and inserted in their places, in order that, if error be committed in any of the columns, it may be arrested, and not suffered to proceed beyond the next point of verification.

It may be pointed out that the last horizontal series,—that corresponding to $x=97$,—which will serve as a final verification of all the columns, may be formed most conveniently without the aid of the machine, as follows:—When $x=97$, the expression for the value of the policy becomes,

$$V_{w}]_{97}=1-(1+a_w)^{-1};$$

and the series of final terms will be formed in order, by subtracting from unity, continuously, $(1+a_{10})^{-1}$ and the differences of the series of which this is the first term. Thus:—

	$V_{w}]_{97}$
	*
	1·0000050
	0397640
10	·9602410
	2435
11	·9599975
	2920
12	·9597055
	3339
13	·9593716
	3675
14	·9590041
	3931
15	·9586110
	4094
16	·9582016
	4156
17	·9577860
*	*

The last term of this series ($x=97$, $w=97$), like those of all the other horizontal series, will come out equal to 0.

There is another arrangement of the table of the Values of Policies which is usually preferred. In it the values of w are still at the top; but the argument at the side, instead of x, is

$x-w$, the duration of the policy. The following is the commencement of the table according to this arrangement.

$x-w$	10	11	12	13	14	15	$x-w$
0	·00000	·00000	·00000	·00000	·00000	·00000	0
1	·00609	·00725	·00822	·00896	·00950	·00979	1
2	·01329	·01540	·01711	·01837	·01920	·01954	2
3	·02140	·02423	·02644	.02799	·02885	·02899	3
4	·03017	·03349	·03598	·03755	·03821	·03788	4
5	·03938	·04296	·04546	·04683	·04702	·04618	5
6	·04879	·05238	·05467	·05556	·05524	·05421	6
7	·05815	·06152	·06332	·06370	·06320	·06217	7
8	·06723	·07011	·07140	·07159	·07108	·07031	8
9	·07577	·07813	·07922	·07940	·07914	·07882	9
10	·08374	·08589	·08697	·08739	·08757	·08776	10
*	*	*	*	*	*	*	*

The computation can be conducted in this form just as easily as in the other. The only changes in the process will be, that the slip containing the differences, while being carried forward from each column to the next, will have to be raised one line; and that the series formed for verification will take their places, not in horizontal lines, but in ascending diagonal lines.*

Problem III.—To construct Columns N_x and D_x, l_x and v^x being given.

It is advisedly that I here place N_x before D_x, the construction in this order being the easier of the two.

We have, $N_x = D_{x+1} + D_{x+2} + \ldots ;$

whence, $\Delta N_x = -D_{x+1} = -v^{x+1} l_{x+1},$

and, $N_{x+1} = N_x + \Delta N_x = N_x - v^{x+1} l_{x+1}.$

By aid of this expression, which is of the form $P-QR$, the column might be constructed, if we had the means of determining independently N_x for the youngest tabular age, namely 10 years. But this we have not. We therefore commence with the oldest age, and writing the above

$$N_x = N_{x+1} + v^{x+1} l_{x+1},$$

we have for an initial term, N_{97} being $=0$,

$$N_{96} = v^{97} l_{97}.$$

The method of forming this is obvious.

* The process of this problem is very clearly described in a paper by Dr. Zillmer in vol. xv, p. 25, of this *Journal*.

For the remainder of the column, the expression being now of the form, P+QR, we have $P=N_{x+1}$, $Q=v^{x+1}$, and $R=l_{x+1}$; and the manner of proceeding here, too, is obvious.

It will be observed, however, that the operation here differs in an important respect from those of the preceding problems. In them Q (the *in* factor) being constant, the setting on F remains unchanged till the completion of the column; here, however, $Q(=v^{x+1})$ varies with x, and the setting has to be altered for each term. The operation is nevertheless still continuous, inasmuch as each result enters into, and forms part of, that which follows.

The following is a specimen of the process. It includes also the formation of D_x.

	v^x			
	520 3284			
x	535 9383		N_x	D_x
97	552 0164	l_x	·00000000	·51171921
6	568 5769	9	·51171921	2·86960758
5	585 6342	49	3·38132679	8·14324320
4	603 2032	135	11·52456999	17·02360082
3	621 2993	274	28·54817081	30·01310627
2	639 9383	469	58·56127708	47·65556172
1	659 1364	723	106·21683880	71·42138460
0	678 9105	1052	177·63822340	102·09457340
*	* *	*	* * *	* * *
59	1697 3309	58866	112263·88510	10582·68657
8	1748 2508	60533	122846·57167	11186·83881
7	1800 6984	62125	134033·41048	11805·65929
6	1854 7193	63652	145839·06977	12439·12396
5	1910 3609	65114	158278·19373	13087·57478
4	1967 6717	66513	171365·76851	13751·57773
3	2026 7019	67852	185117·34624	14432·57755
2	2087 5029	69138	199549·92379	15131·09578
1	2150 1280	70373	214681·01957	15849·23394
0	2214 6318	71566	230530·25351	16589·31550
*	* *	*	* * *	* * *
19	5536 7575	96223	1227620·99017	55191·71170
8	5702 8603	96779	1282812·70187	57121·18885
7	5873 9461	97245	1339933·89072	59064·12591
6	6050 1645	97624	1398998·01663	61034·21644
5	6231 6694	97942	1460032·23307	63046·24818
4	6418 6195	98224	1523078·48125	65117·45981
3	6611 1781	98496	1588195·94106	67267·09717
2	6809 5134	98784	1655463·03823	69515·86405
1	7013 7988	99113	1724978·90228	71888·13958
0	7224 2126	99510	1796867·04186	74409·39100
	7440 9391	100000	1871276·43286	

The two columns following that containing the ages, represent two cardboard slips, on which are written, in reverse order, the terms of v^x and l_x, respectively. Their position with respect to each other and to the ages in the margin, is regulated in accordance with the formula of construction. Here we have, for example, opposite $x=96$, v^{97} and l_{97}; opposite $x=95$, v^{96} and l_{96}, and so on. The process now consists in the multiplying together of the corresponding numbers on the cards, in succession, and the setting down of the results, as they arise, on the same line. These, as stated, are the terms of N_x corresponding to the values of x in the margin. It will be understood that in this process, in accordance with the formula of construction, the results are not removed from the machine, (S_2) alone requiring to be effaced after each multiplication.

I say nothing here as to the method of procuring verification of the foregoing process. I defer this, with further remarks on the construction, till the next following problem shall have been discussed.

Next, to form D_x. We have, as above,

$$D_x=-\Delta N_{x-1}=N_{x-1}-N_x.$$

Hence the Column D_x will be formed by differencing, in the usual way, the Column N_x, when written as above in reverse order. And the differences, when written as in the specimen, each opposite the subtrahend by the employment of which it is deduced, will be in their proper relation to the ages in the margin. And we have now, consequently, in line with x, v^{x+1}, l_{x+1}, N_x, and D_x.

To form D_x independently of N_x, the same process as that for the formation of the latter column would have to be gone through, with the addition that besides (S_2), (S_1) also would have to be effaced after each multiplication. The process also would be discontinuous; and in consequence, unless the work were performed in duplicate, no verification could be procured till both N_x and M_x should be in course of formation. If to this it be added that the deduction of D_x from N_x is at least as easy as that of N_x from D_x, sufficient will have been said to make it manifest that, in the construction of a Commutation Table by the aid of the Arithmometer, the proper course, in regard to the annuity columns, is to commence the formation with N_x. In the case of the assurance columns also, corresponding advantages attend the commencement of the formation with M_x.

Problem IV.—Given v^x and d_x; to construct Columns M_x and C_x.

Since, $$M_x = C_x + C_{x+1} + \dots.$$

∴ $$\Delta M_x = -C_x = -v^{x+1} d_x.$$

Hence, $$M_{x+1} = M_x + \Delta M_x = M_x - v^{x+1} d_x.$$

Column M_x, like Column N_x, and for a similar reason, must be constructed in reverse order. Accordingly, transposing, we have,

$$M_x = M_{x+1} + v^{x+1} d_x,$$

a formula for the construction in such entire accordance with that of the last problem, that little more is necessary than to direct attention to the specimen here given.

x	v^x	d_x	M_x	C_x
	520 3284			
	535 9383		·00000000	
97	552 0164	9	·49681476	·49681476
6	568 5769	40	2·77112236	2·27430760
5	585 6342	86	7·80757648	5·03645412
4	603 2032	139	16·19210096	8·38452448
3	621 2993	195	28·30743731	12·11533635
2	639 9383	254	44·56187013	16·25443282
1	659 1364	329	66·24745769	21·68558756
0	678 9105	408	93·94700609	27·69954840
*	* *	*	* * *	* * *
59	1697 3309	1667	7004·63125	282·94506
8	1748 2508	1592	7282·95278	278·32153
7	1800 6984	1527	7557·91943	274·96665
6	1854 7193	1462	7829·07939	271·15996
5	1910 3609	1399	8096·33888	267·25949
4	1967 6717	1339	8359·81012	263·47124
3	2026 7019	1286	8620·44398	260·63386
2	2087 5029	1235	8878·25059	257·80661
1	2150 1280	1193	9134·76086	256·51027
0	2214 6318	1160	9391·65815	256·89729
*	* *	*	* * *	* * *
19	5536 7575	556	17828·23462	307·84372
8	5702 8603	466	18093·98791	265·75329
7	5873 9461	379	18316·61047	222·62256
6	6050 1645	318	18509·00570	192·39523
5	6231 6694	282	18684·73878	175·73308
4	6418 6195	272	18859·32523	174·58645
3	6611 1781	288	19049·72716	190·40193
2	6809 5134	329	19273·76015	224·03299
1	7013 7988	397	19552·20796	278·44781
0	7224 2126	490	19906·19438	353·98642

The cards here are v^x and d_x, and their disposition, to accord with the working formula, is somewhat different from that of the cards in the last problem. Thus, in line with age x we have here v^{x+1} and d_x. When so arranged, the process for the formation of M_x is identical with that of last problem for the formation of N_x.

To form C_x,

$$C_x = -\Delta M_x = M_x - M_{x+1}.$$

Hence the column will be formed by differencing, as they stand, the terms of M_x; observing that here the differences as they arise are to be written, not as in the last problem, opposite the subtrahends, but opposite the minuends. When so written we have, in line with x, v^{x+1}, d_x, M_x, and C_x.

It has not been usual hitherto to tabulate C_x. But this column can be turned to such good account, for the formation of various subsidiary tables by the aid of the Arithmometer, that it will probably come to be tabulated, in a special category, along with others similarly adapted, such as Δa_x, D_x^{-1}, N_x^{-1}, &c.

When we multiply together two factors, of which one is a non-terminating number, a portion of the result has to be rejected, as not properly belonging to the product; since it would be altered if the non-terminating number were further extended. And therefore it is that in the process of contracted multiplication labour is saved by so arranging the work that only the *correct* portion of the product is formed. The Arithmometer attains the same end in another way: in the use of it we form the *entire* product of the numbers submitted to it, and we neglect the useless figures in recording the results. It is therefore desirable to be furnished with a guide as to the extent to which this process of curtailment ought to be carried.

In the case supposed—the multiplication of a terminating and a non-terminating factor—we know that we can depend upon about as many figures in the product as there are of significant figures in the non-terminating factor; and hence in the several terms of D_x and C_x, we shall have as many places true as there are in the terms of v^x which enter them, respectively. Also, in the several terms of N_x and M_x, which are respectively summations of series of terms in D_x and C_x, we should expect to find one or two more figures correct than in the corresponding terms of D_x and C_x.

We may say, then, that in the formation of N_x and M_x it is unnecessary to record any term to more than two places beyond the number of significant figures in the power of v in immediate use.

We shall find these conclusions verified in the case of the columns before us. I have used in the formation the column v^x (3 per-cent) as contained in Jones, vol. i, pp. 79 and 82, having first verified it by the aid of the Arithmometer. It contains seven significant figures at the outset, which further on are increased to eight; and I have used it to its full extent for experiment, although generally one or two places fewer will be considered sufficient.

Corresponding values in N_x and M_x are connected by the equation,

$$M_x = vN_{x-1} - N_x;$$

and applying this at a few points we find,

$$\begin{aligned} M_{91} &= 66{\cdot}24745{,}576 \\ M_{59} &= 7004{\cdot}631{,}09 \\ M_{19} &= 17828{\cdot}234{,}95 \\ M_{10} &= 19906{\cdot}1939{,}3 \end{aligned}$$

The commas indicate the extent to which the values thus formed agree with those formed by the use of the machine. The agreement in the four examples extends to seven, eight, and nine places, respectively; but when we attend to the manner in which the formula lends itself to these deductions, we shall not fail to see that in each case the correctness of the Ns concerned is established generally to two places more. For example :—

103)128281270187	N_{18}
124544922512	vN_{18}
122762099017	N_{19}
17828234,95	M_{19}

Ten places in N_{18} and N_{19} we perceive here come into use for the determination of the eight correct places in M_{19}. We conclude, therefore, that at this part of the table N_x may be depended on to ten places of figures, and no more than this number need have been recorded.

Also, if the work has been correctly performed, there appears no reason why M_x should not be accepted as true to the same extent as N_x. As to the number of places to be finally tabulated for use, the computer will, of course, follow the dictates of his own judgment.

By the use of the foregoing formula, N_x and M_x may be verified as the work proceeds. Both columns being brought down to the same point by the formation of, say, twenty terms in each, comparison can be made at this point as above shown. If found satisfactory, the formation may be continued till another point of comparison is reached; and so on till the columns are completed.

It is hardly necessary to point out that the contrivance formerly suggested for correcting the last figure retained in the several terms, cannot in these constructions, be advantageously applied; the reason being, that here the last figures vary in their distances from the decimal point.

The pegs also are employed here in a manner different from that in which they came into use in Problems I and II. They are now employed to facilitate the recording, by simply separating the results on S_1 into convenient periods.

Problem V.—To construct a complete table of the Values of Temporary Annuities; that is, a table which shall comprise all ages and all durations.

This benefit, being a function of the ages at commencement and cessation, say x and w (to the former of which ages also its value is referred), it will be here for convenience denoted, in accordance with the usual functional notation, by $\phi(x, w)$; and its duration, consequently, will be $w-x=n$.

Its value is

$$\phi(x, w)=\frac{N_x-N_w}{D_x}=D_x^{-1}.(N_x-N_w) \quad . \quad . \quad . \quad (6)$$

The work of construction will be arranged as in the example following, in columns, having at the top the successive values of x, and at the side those of w. Hence w will vary in the columns, and x in the rows.

The formation in columns being the more convenient, let, first, w vary. Then we have from (6),

$$\Delta_w\phi(x, w)=-D_x^{-1}.\Delta N_w=D_x^{-1}.D_{w+1};$$

whence, $\quad \phi(x, w+1)=\phi(x, w)+D_x^{-1}.D_{w+1}.$

This is of the form $P+QR$, $Q(=D_x^{-1})$ being constant. By the aid of this expression the columns will be successively formed.

		134392	139105	143852	148661	153569	158614
		10	11	12	13	14	15
w	D_w						
10	744094	·0000					
11	718881	·9661	·0000				
12	695159	1·9004	·9670	·0000			
13	672671	2·8044	1·9027	·9677	·0000		
14	651175	3·6795	2·8085	1·9044	·9680	·0000	
15	630462	4·5268	3·6855	2·8113	1·9053	·9682	·0000
16	610342	5·3470	4·5346	3·6893	2·8126	1·9055	·9681
17	590641	6·1408	5·3562	4·5389	3·6907	2·8125	1·9049
18	571212	6·9085	6·1508	5·3606	4·5399	3·6897	2·8109
19	551917	7·6502	6·9185	6·1546	5·3603	4·5373	3·6864
20	532763	8·3662	7·6596	6·9210	6·1524	5·3555	4·5314
*	*	*	*	*	*	*	*

The process is as follows:—The successive values of D_x^{-1} $(=1\div D_x)$ are written in order, each at the top of its proper column, and a card containing a transcript of Column D is placed at the side, as shown. Then, observing that, as appears from (6), when $x=w$, $\phi(x, w)=0$, this may be at once inserted as the initial term in all the columns. Now, set $D_{10}^{-1}=134392$ on F, and multiply successively by D_{11}, D_{12}, &c.; the results will be the terms in Column 10 opposite the ages 11, 12, &c. And so Column 10 will be filled up.

Moving the card forward into connexion with Column 11, setting the reciprocal at the top of the column upon F, and multiplying successively by D_{12}, &c., this column will be filled up in like manner. And so the columns may be filled up in succession, till the table is completed.

The method of construction here may be looked at in a somewhat different manner. The terms occupying, say, the column headed x, are the values of annuities on (x) to last one year, two years, three years, &c.; and they are respectively equal to

$$\text{One year, } \frac{D_{x+1}}{D_x},$$

$$\text{Two years, } \frac{D_{x+1}}{D_x}+\frac{D_{x+2}}{D_x},$$

$$\text{Three years, } \frac{D_{x+1}}{D_x}+\frac{D_{x+2}}{D_x}+\frac{D_{x+3}}{D_x},$$

and so on, the annuity for n years being formed by adding another term, $\frac{D_{x+n}}{D_x}$, to that for $n-1$ years. And the process described is nothing else than the orderly formation of these terms, and the adding of each, as it is formed, to the preceding result.*

The intelligent computer, also, will not fail to observe that, as the additions made in passing from each value to the next consist at the outset of no more than four places, which number becomes reduced also, as we descend the columns, to three, two, and one, it would be a waste of power to continue to employ six places in the factors by the multiplication of which they are produced. It must be left to his own judgment to determine the points at which, and the extent to which, curtailment of the factors can with safety be made. The verifications will serve as a check on excessive curtailment.

Verification will be obtained, as in other cases, by the separate formation of a few rows at suitable intervals.

When x varies, we have, from (6),

$$\Delta_x\phi(x, w) = \Delta . (D_x^{-1}N_x) - N_w . \Delta D_x^{-1},$$
$$= \Delta a_x - N_w . \Delta D_x^{-1};$$

whence, $\phi(x+1, w) = \phi(x, w) + \Delta a_x - N_w . \Delta D_x^{-1}.$

This expression is unsuited for our purpose; it does not give a continuous process fitted for the application of the Arithmometer. We must, therefore, arrange for the independent formation of each term.

Resuming (6), the successive values of the numerator, when x varies, are easily formed, as follows:—

$$\Delta_x(N_x - N_w) = -D_{x+1};$$

whence, $(N_{x+1} - N_w) = (N_x - N_w) - D_{x+1}.$

So that, forming this function for $x=10$ and any constant value of w, say 20, the succeeding values will be formed by the continuous subtraction of the terms of D_x in order, commencing with D_{11}. Then, writing opposite the terms thus formed the reciprocals D_{10}^{-1}, &c., also in order, multiplication of the corresponding terms, by the machine, gives the required results.

* See General Hannyngton's paper, already referred to, p. 252.

The following is the whole of the work for this row:—

	$N_x - N_{20}$	D_x^{-1}	$\phi(x, 20)$
N_{10}	1796867		
N_{20}	1174345		
10	622522	134392	8·3662
	71888		
11	550634	139105	7·6596
	69516		
12	481118	143852	6·9210
	67267		
13	413851	148661	6·1524
	65118		
14	348733	153569	5·3555
	63046		
15	285687	158614	4·5314
	61034		
16	224653	163843	3·6808
	59064		
17	165589	169308	2·8036
	57121		
18	108468	175066	1·8989
	55192		
19	53276	181187	0·9653
	53276		
20	00000	187701	0·0000

The formation of three or four rows in this manner will be sufficient, the values formed being inserted in their places before commencing the construction in the columns.

Problem VI.—To construct a complete table of the Values of Deferred Annuities.

This benefit, like the last, being a function of two ages, namely, the present age, x, and that at which the annuity is to be entered on, w, it will be, for our present purpose, denoted by $\psi(x, w)$.

We have, then,

$$\psi(x, w) = \frac{N_w}{D_x} = D_x^{-1} . N_w \quad . \quad . \quad . \quad . \quad (7)$$

First let w vary, and we have,

$$\Delta_w \psi(x, w) = D_x{}^{-1} . \Delta N_w = -D_x{}^{-1} . D_{w+1} .$$

Whence, $\psi(x, w+1) = \psi(x, w) - D_x{}^{-1} . D_{w+1}$ . . . (8)

By this formula the required table may be constructed, the values in which x, the present age, is constant, taking their places in the columns, and those in which w, the age at entry, is constant, taking their places in the rows.

The following is a specimen of the commencement of the table.

		134392	139105	143852	148661	153569	158614
w	D_w	10	11	12	13	14	15
10	74409	24·1485					
11	71888	23·1823	23·9953				
12	69516	22·2481	23·0283	23·8142			
13	67267	21·3441	22·0926	22·8465	23·6103		
14	65118	20·4689	21·1868	21·9098	22·6422	23·3898	
15	63046	19·6217	20·3098	21·0029	21·7050	22·4216	23·1581
16	61034	18·8014	19·4608	20·1249	20·7976	21·4843	22·1901
17	59064	18·0076	18·6392	19·2752	19·9196	20·5772	21·2532
18	57121	17·2400	17·8446	18·4535	19·0704	19·7000	20·3472
19	55192	16·4982	17·0768	17·6596	18·2499	18·8525	19·4718
20	53276	15·7823	16·3357	16·8932	17·4579	18·0343	18·6267
*	*	*	*	*	*	*	*

The numbers at the tops of the columns are, as in the last problem, the successive values of $D_x{}^{-1}$, corresponding to the ages over which they are placed, respectively; and the column on the left, represents a card containing the successive values of D_w.

In the initial term of each column $x = w$; so that these terms are, in fact, the whole-life annuities at the ages under and opposite which they are found.

For Column 10 we have for the initial term,

$$\psi(10, 10) = D_{10}{}^{-1} N_{10} .$$

Hence, taking $D_{10}{}^{-1}(=134392)$ for the in-factor, and, with the regulator at addition, multiplying by $N_{10}(=1796867)$, we get for the first value in this column 24·1485. The regulator is now reversed, in accordance with (8), and multiplying by D_{11}, D_{12}, &c., the remaining values in this column are successively produced.

For Column 11 the initial term is $D_{11}{}^{-1} N_{11}, = 139105 \times 1724979$, and the successive multipliers are D_{12}, D_{13}, &c.; and so on, throughout the remaining columns, the process in all being similar.

The card is moved forward as the work proceeds, being always placed against the column in course of formation.

Values for verification are obtained as follows. Let, in (7), x vary, and we have,

$$\Delta_x \psi(x, w) = \mathrm{N}_w . \Delta \mathrm{D}_x^{-1};$$

whence, $$\psi(x+1, w) = \psi(x, w) + \mathrm{N}_w . \Delta \mathrm{D}_x^{-1}.$$

By this formula we should be able to form in succession the values occupying any of the rows; for it is in these that x varies while w is constant.

The following shows the formation of the row corresponding to $w=20$.

x	$\Delta \mathrm{D}_x^{-1}$	$\psi(x, 20)$
10		15·7823
11	4713	16·3357
12	4747	16·8932
13	4809	17·4579
14	4908	18·0343
15	5045	18·6268
16	5229	19·2408
17	5465	19·8826
18	5758	20·5588
19	6121	21·2776
20	6514	22·0426
*	*	*

The initial term here is $\psi(10, 20) = \mathrm{N}_{20}\mathrm{D}_{10}^{-1}$; and N_{20} being constant it is taken for the in-factor, and set upon F. Multiplying now by $\mathrm{D}_{10}^{-1}(=134392)$ and the differences of the series of which it is the first term (here represented as written on a card), and the regulator being at addition, the required values are produced in succession, as shown. Verification of this series is had in the visible correspondence of its last term, $\psi(20, 20)$, with a_{20}.

The formation of three or four of these horizontal series will be sufficient, the terms of which should be inserted in their places before commencing the construction in the columns.

It will be observed that only five places have been used in the terms of D_w employed in the process. The terms of D_x^{-1}, the other factor used in the formation of the addends, has six significant figures; and the product of these two can certainly be depended on to five places, while four is the utmost extent of the addends in the columns. An effect of the restriction of the out-factors, when it can be done with safety, is, of course, a saving of time and labour.

Since $\phi(x, w) + \psi(x, w) = a_x$, it follows that, in consequence of the arrangement adopted in the formations of Problems V and VI, the sum of corresponding values in the two formations is equal to the value of the whole-life annuity at the age under which the two values are found. Thus, in Column 13, and opposite 17, we have, for example, $3·6907 + 19·9196 = 23·6103, = a_{13}$. And in this manner any of the values formed may be checked.

The results of the two problems admit of arrangement in one table, as follows:—

	10	11	12	13	14	15	
10	24·1485	23·1823	22·2481	21·3441	20·4689	19·6217	10
1	·9661	23·9953	23·0283	22·0926	21·1868	20·3098	1
2	1·9004	·9670	23·8142	22·8465	21·9098	21·0029	2
3	2·8044	1·9027	·9677	23·6103	22·6422	21·7050	3
4	3·6795	2·8085	1·9044	·9680	23·3898	22·4216	4
15	4·5268	3·6855	2·8113	1·9053	·9682	23·1581	15
6	5·3470	4·5346	3·6893	2·8126	1·9055	·9681	6
7	6·1408	5·3562	4·5389	3·6907	2·8125	1·9049	7
8	6·9085	6·1508	5·3606	4·5399	3·6897	2·8109	8
9	7·6502	6·9185	6·1546	5·3603	4·5373	3·6864	9
20	8·3662	7·6596	6·9210	6·1524	5·3555	4·5314	20
*	*	*	*	*	*	*	*

The deferred annuities occupy the upper part of the table, the values of x, the present age, being at the side, and those of w, the ages at which the annuities are respectively to be entered on, at the top. The temporary annuities occupy the portion of the table below the diagonal, and as regards them the positions of the values of x and w are reversed, those of the former being at the top, and those of the latter, which are the ages at which the annuities are respectively to cease, at the side. The diagonal is legitimately occupied by the whole-life annuities, the indications connected with which are to be read in conformity with those of the deferred annuities. Thus, under 14 and opposite 14 we have $23·3898 = a_{14}$, the annuity on (14), commencing at 14.*

Problem VII.—To construct Columns $D_{x.y}$ and $N_{x.y}$.

To effect these constructions we may commence with either of the two functions specified, and thence deduce the other by a simple operation.

* See the *Institute Tables*, Introduction, p. xxxv.

First. To commence with $D_{x.y}$. If x is equal to or greater than y, we have,

$$D_{x.y} = l_x l_y v^x = D_x l_y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

Let y vary; then,

$$\Delta_y D_{x.y} = D_x . \Delta l_y = -D_x d_y,$$

since d_y is negative. Hence,

$$D_{x.y+1} = D_{x.y} - D_x d_y.$$

It is most advantageous to form the series in reverse order. Therefore, transposing,

$$D_{x.y} = D_{x.y+1} + D_x d_y.$$

Or, changing y into $y-1$,

$$D_{x.y-1} = D_{x.y} + D_x d_{y-1} \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

which is of the form $P+QR$.

The work will be first arranged in columns, in which x is constant, and y consequently variable. And to facilitate subsequent operations it is desirable that values of the function in which $x-y$ is constant should occupy the same row. We have therefore, from (9), for the initial terms in the successive columns, $D_{97}l_{97}$, $D_{96}l_{96}$, $D_{95}l_{95}$, &c.; and (10) is the working formula for the completion of the several columns.

The following is a specimen of the work, showing the commencement of the first four columns.

		5117	28696	81432	170236
$x-y$	d_y	97	96	95	94
0	9	·046053	1·406104	10·993320	46·644664
1	40	·250733	3·873960	22·312368	79·840684
2	86	·690795	7·862704	38·191608	123·080628
3	139	1·402058	13·458424	58·875336	179·088272
4	195	2·399873	20·747208	85·666464	248·544560
5	254	3·699591	30·188192	118·890720	332·811380
6	329	5·383084	41·896160	159·199560	437·506520
7	408	7·470820	56·100680	209·280240	569·098948
8	495	10·003735	73·748720	272·227176	729·291024
9	615	13·150690	95·930728	348·854688	923·019592
10	773	17·106131	122·933664	441·524304	1152·157248
*	941	*	*	*	*
	1138				
	1346				
	*				

The values of x are, as stated, at the top, each in connexion with the column in which it is constant, and over these are placed the values of D_x. The values of $x-y$ are at the side, each being constant in the row opposite which it is placed. The values of d_y, in reverse order, are on a card represented as applied to Column 97. The first step is, setting $D_{97}=5117$ upon F, to multiply by $9=l_{97}$, which gives $46053=D_{97.97}$. And the column is completed by using continuously as multipliers the successive terms of d_y, commencing with 40, the second term in this order.

The card is now moved forward to the next column, and at the same time upwards one line; and by a similar manner of proceeding this column, too, will be filled up. And so on with the other columns in succession.

I have pointed the values formed in the specimen in accordance with a radix of 10,000. In consequence, the necessity for the exhibition of many useless figures, in the values corresponding to the younger ages, will be avoided. Thus we should now have $D_{10.10}=744094000$, instead of 74409400000.

Verification is obtained as follows:—Let, in (9), x vary, and we have,

$$\Delta_x D_{x.y}=l_y.\Delta D_x;$$

whence,

$$D_{x+1.y}=D_{x.y}+l_y.\Delta D_x.$$

Transposing, and writing $x-1$ for x,

$$D_{x-1.y}=D_{x.y}-l_y.\Delta D_{x-1}.$$

The series here indicated, in which y is constant and x decreases by a unit in passing from term to term, ascend diagonally in the preceding formation, as a little consideration will show. And the construction of three or four of them, at intervals, by the aid of the formula just deduced, will suffice for the verification of the whole work.

The following is the formation of a portion of the series commencing with $D_{97.87}$ $(x-y=10)$. It must be observed that ΔD_x being negative, the regulator will be set for addition.

$x-y$	ΔD_x	87
10		17106131
9	23579	95930728
8	52736	272227176
7	88804	569098948
6	129895	*
*	*	

Here $l_y = l_{87} = 3343$, being constant, it is taken for the *in* factor, and set upon F. Successive and continuous multiplication by $D_{97} = 5117$, and the terms of ΔD_x in reverse order, will then complete the series, which will consist of eleven terms, terminating with $D_{87.87}$.

The following is the commencement of the formation of the first four columns of $N_{x.y}$.

x	Diff. 0.	Diff. 1.	Diff. 2.	Diff. 3.
96	·046053 1·406104	·250733 3·873960	·690795 7·862704	1·402058 13·458424
95	1·452157 10·993320	4·124693 22·312368	8·553499 38·191608	14·860482 58·875336
94	12·445477 46·644664	26·437061 79·840684	46·745107 123·080628	73·735818 179·088272
93	59·090141	106·277745	169·825735	252·824090
*	*	*	*	*

The terms of $D_{x.y}$ occupying the rows in the last formation are transcribed in columns, as shown; and by continuous addition the terms of $N_{x.y}$ are formed. The values of x only are inserted. These are constant in the rows; and so the value of y in any particular term is determined by subtracting the Diff. at the top of the column from the value of x pertaining to the row in which the term in question is found.

There is another method of arranging the results in the formation of $D_{x.y}$, by the employment of which the labour of transcribing them for the formation of $N_{x.y}$ is saved. The arrangement is that adopted by General Hannyngton in the example in his paper of the construction of $D_{x.y}$. In it the successive results, which in the preceding formation are written in columns, are written in rows, commencing in all cases in the first column. In consequence, the terms of $D_{x.y}$ in which $x-y$ is constant occupy the same column; and if the rows are written, not on succeeding but on alternate lines, we are prepared immediately, on the completion of $D_{x.y}$, for the formation of $N_{x.y}$.

The following is an example.

x	D_x	Diff. 0.	Diff. 1.	Diff. 2.	Diff. 3.
97 96	5117 28696	·046053 1·406104	·250733 3·873960	·690795 7·862704	1·402058 13·458424
95	81432	1·452157 10·993320	4·124693 22·312368	8·553499 38·191608	14·860182 58·875336
94	170236	12·445477 46·644664	26·437061 79·840684	46·745107 123·080628	73·735818 179·088272
93	300131	59·090141 140·761439	106·277745 216·994713	169·825735 315·737812	252·824090 438·191260
92	476556	199·851580 344·549988	323·272458 501·336912	485·563547 695·771760	691·015350 931·666980
91	714214	544·401568 751·353128	824·609370 1042·752440	1181·335307 1396·288370	1622·682330 1835·529980
90	102095	1295·754696 1490·58700	1867·361810 1995·95725	2577·623677 2623·84150	3458·212310 3413·03585
89	140810	2786·34170 2752·83550	3863·31906 3618·81700	5201·46518 4707·27830	6871·24816 6032·30040
88	190659	5539·17720 4899·93630	7482·13606 6373·73037	9908·74348 8167·83156	12903·54856 10337·53098
87	255445	10439·11350 8539·52635	13855·86643 10943·26380	18076·57504 13850·22790	23241·07954 17288·51760
		18978·63985	24799·13023	31926·80294	40529·59714
*	*	*	*	*	*

The values of x at the side are those that belong to the terms of $D_{x.y}$ in the rows opposite which they are placed; and the values of the same symbol belonging to the terms of $N_{x.y}$, if inserted opposite the rows in which these values are constant, would be each a unit less than that occupying the line above it, observing, however, that $D_{97.97} = N_{96.96}$.

But, secondly, we may commence the construction with $N_{x.y}$, whence it is easy to pass to $D_{x.y}$.

Since, $N_{x.y} = D_{x+1.y+1} + D_{x+2.y+2} + \ldots$ therefore, if x and y both vary,

$$\Delta_{x.y} N_{x.y} = -D_{x+1.y+1} = -D_{x+1} l_{y+1}.$$

Whence, $N_{x+1.y+1} = N_{x.y} - D_{x+1} l_{y+1}$.

Transposing, as the series must be formed in reverse order,

$$N_{x.y} = N_{x+1.y+1} + D_{x+1} l_{y+1};$$

or

$$N_{x-1.y-1} = N_{x.y} + D_x l_y.$$

And by this formula, which corresponds to $P+QR$, where both Q and R vary in passing from each term to the next, the table may be constructed.

The following is a specimen, showing the commencement of the first four columns.

x	D_x	l_y	Diff. 0.	Diff. 1.	Diff. 2.	Diff. 3.
96	5117	9	·046053	·250733	·690795	1·402058
95	28696	49	1·452157	4·124693	8·553499	14·860482
94	81432	135	12·445477	26·437061	46·745107	73·735818
93	170236	274	59·090141	106·277745	169·825735	252·824090
92	300131	469	199·851580	323·272458	485·563547	691·015350
91	476556	723	544·401568	824·609370	1181·335307	1622·682330
90	714214	1052	1295·754696	1867·361810	2577·623677	3458·212310
89	102095	1460	2786·34170	3863·31906	5201·46518	6871·24816
88	140810	1955	5539·17720	7482·13606	9908·74348	12903·54856
87	190659	2570	10439·11350	13855·86643	18076·57504	23241·07954
86	255445	3343	18978·63985	24799·13023	31926·80294	40529·59714
*	*	4284	*	*	*	*
		5422				
		6768				
		*				

It will be at once perceived that the process here, as regards the individual columns, is analogous to that of the formation of N_x in Problem III. The card containing v^x there, is here replaced by one containing D_x; and the successive operations are, in both, the multiplying together of the two factors in line with each other. On the completion of each column the cards are moved forward into connexion with the next, the one containing l_x being at the same time raised one line, so as to preserve the proper relation as to difference of ages between the factors that come together.

In practice, the alternate columns will be left blank, to receive the terms of $D_{x.y}$, which will be formed by differencing, as in the example of Problem III. This need not be exemplified here.

I am not prepared to say absolutely to which of the two methods that have been explained the preference ought to be awarded. I incline at present to prefer the first, that in which we commence with $D_{x.y}$, arranging the results as suggested by General Hannyngton. This method is certainly attended with less labour and risk of error than the other, inasmuch as in it the setting on F has to be changed only once for each column, while when the second method is used it has to be changed for each value formed. In the first method, also, the terms of $D_{x.y}$ admit of exact verification, as I have shown, and it is easy to secure the correctness of the terms of $N_{x.y}$ by preliminary addition of the terms of $D_{x.y}$ in groups; while on the other hand, when the second method is employed, there is no other way of verifying the work than by repetition of the process.

A drawback in the use of the first method is that, unless an assistant is available, it is less convenient to record the successive results in rows than in columns. This drawback probably will not, in the circumstances, be deemed of much importance.

A more extended experience than has yet been acquired will be necessary to determine the question authoritatively.

In the examples under the present problem the values formed are given to their full extent. But it must be understood that those of $D_{x.y}$ have no claim to be true to more than six places, and those of $N_{x.y}$ to more than seven, perhaps eight. If this extent shall be judged insufficient, it will be no difficult matter, with the means now available, to form a new Column D_x to seven places, for special use in the formation now under consideration.

I may, perhaps, be allowed here to say, that the first of the foregoing methods of construction, apart from General Hannyngton's manner of arranging the results, is substantially one that I developed and practised upwards of twenty years ago.* It is perfected by the employment of the Arithmometer in the formation of the addends.

Problem VIII.—To construct a table for the formation of Survivorship Assurances.

A survivorship assurance on (x) against (y) is an assurance payable at the end of the year in which the combination $(x.y)$ is dissolved, provided the dissolution is caused by the death of (x).

The value of this assurance, in respect of the nth year, is the value of the sum payable (supposed a unit) into the probability of the concurrence of these two events, namely, that (x) shall die in that year, and that (y) shall live over half of it. This gives

$$\frac{d_x \lambda_y v^n}{l_{x.y}},$$

in which $\lambda_y = \frac{1}{2}(l_y + l_{y+1})$, the number-living at age $y+\frac{1}{2}$, on the hypothesis of uniform distribution of the deaths of each year. Summation of this expression, for the proper values of n, will give the value of the assurance with reference to the whole or any portion of the future lifetime of (x).

Making $n=1$, and multiplying numerator and denominator by v^x for the compartment of the table in which x is not less than y, and by v^y for that in which y is not less than x, we have the following expressions for the value in respect of the first year in the two cases:—

$$\frac{d_x \lambda_y v^{x+1}}{D_{x.y}}, \text{ and } \frac{d_x \lambda_y v^{y+1}}{D_{x.y}}.$$

Consequently, $D_{x.y}$ having been already formed, we have now to form only the numerators of these expressions, of the first for every combination of x and y in which x is not less than y, and of the second for every combination in which y is not less than x.

The following is the commencement of the two series d_x and

* See *Tables and Formulæ for the Computation of Life Contingencies*, (1849,) pp. 149 to 181; and *Assurance and Annuity Tables*, (1851,) Introduction, p. x.

λ_y, with their differences. The former is given in the mortality table, and the latter is formed in accordance with the relation $\lambda_y = \frac{1}{2}(l_y + l_{y+1})$.

x & y	d_x	Δ	λ_y	Δ
97	9		4·5	
6	40	31	29·0	24·5
5	86	46	92·0	63·0
4	139	53	204·5	112·5
3	195	56	371·5	167·0
2	254	59	596·0	224·5
1	329	75	887·5	291·5
0	408	79	1256·0	368·5
89	495	87	1707·5	451·5
8	615	120	2262·5	555·0
7	773	158	2956·5	694·0
6	941	168	3813·5	857·0
5	1138	197	4853·0	1039·5
4	1346	208	6095·0	1242·0
3	1545	199	7540·5	1445·5
2	1719	174	9172·5	1632·0
1	1883	164	10973·5	1801·0
0	2015	132	12922·5	1949·0
*	*	*	*	*

It may be well to remark that, the terms being arranged in reverse order, it is necessary, for the purpose of securing the correspondence of the differences with the ages to which they individually belong, to place them each against the term which constitutes the minuend in its formation. And they are all properly negative.

The formation for each age of the functions $d_x v^{x+1}$ and $\lambda_y v^{y+1}$ is the next step. For this purpose the Arithmometer comes into use. The commencement of the formation is here shewn, but it is so plain as not to stand in need of explanation.

x & y	v^{x+1}	d_x	$d_x v^{x+1}$	λ_y	$\lambda_y v^{y+1}$
97	5520164	9	496815	45	24841
6	5685769	40	2274308	290	164887
5	5856342	86	5036454	920	538783
4	6032032	139	8384524	2045	1233551
3	6212993	195	12115336	3715	2308127
2	6399383	254	16254433	5960	3814032
1	6591364	329	21685588	8875	5849836
0	6789105	408	27699548	12560	8527116
89	6992779	495	34614256	17075	11940170
8	7202562	615	44295756	22625	16295797
7	7418639	773	57346079	29565	21933206
6	7641198	941	71903673	38135	29139709
5	7870434	1138	89565539	48530	38195216
4	8106547	1346	91141226	60950	49409404
3	8349743	1545	129003529	75405	62961237
*	*	*	*	*	*

We are now prepared for the principal formations.

When x is equal to or greater than y, we have, if y vary,

$$\Delta_y d_x \lambda_y v^{x+1} = d_x \lambda_y v^{x+1} . \Delta\lambda_y ;$$

whence

$$d_x \lambda_{y+1} v^{x+1} = d_x \lambda_y v^{x+1} + d_x \lambda_y v^{x+1} . \Delta\lambda_y ;$$

and for the formation in reverse order, by transposition,

$$d_x \lambda_y v^{x+1} = d_x \lambda_{y+1} v^{x+1} - d_x \lambda_y v^{x+1} . \Delta\lambda_y .$$

Finally, changing y into $y-1$,

$$d_x \lambda_{y-1} v^{x+1} = d_x \lambda_y v^{x+1} - d_x \lambda_{y-1} v^{x+1} . \Delta\lambda_{y-1} .$$

By this formula, which is of the form P—QR, the required values are to be constructed; but $\Delta\lambda_{y-1}$ being negative, the regulator has to be set for addition.

The following is a specimen of the process.

$x > y$

x	$d_x v^{x+1}$	λ_y	x	Diff. 0.	Diff. 1.	Diff. 2.	Diff. 3.	Diff. 4.
97	49682	45	97	·02236	·14408	·45707	1·01600	1·84569
		245						
6	227431	290		·65955	2·09236	4·65094	8·44902	13·55483
		630	6	·68191	2·23644	5·10801	9·46502	15·40052
5	503645	920		4·63354	10·29956	18·71044	30·01730	44·69858
		1125	5	5·31545	12·53600	23·81845	39·48232	60·09910
4	838452	2045		17·14634	31·14849	49·9717	74·4126	105·3096
		1670	4	22·46179	43·68449	73·7902	113·8949	165·4087
3	1211534	3715		45·0085	72·2074	107·5236	152·1687	206·8694
		2245	3	67·4703	115·8919	181·3138	266·0636	372·2781
2	1625443	5960		96·8765	144·2582	204·1558	277·5446	367·7567
		2915	2	164·3468	260·1501	385·4696	543·6082	740·0348
1	2168559	8875		192·4595	272·3709	370·2813	490·6362	641·1342
		3685	1	356·8063	532·5210	755·7509	1034·2444	1381·1690
0	2769955	12560		347·9062	472·9696	626·7021	818·9369	1056·322
		4515	0	704·7125	1005·4906	1382·4530	1853·1813	2437·491
89	3461426	17075		591·0385	783·1476	1023·3706	1320·0148	1679·830
		5550	89	1295·7510	1788·6382	2405·8236	3173·1961	4117·321
8	4429576	22625		1002·192	1309·604	1689·219	2149·673	2699·827
		6940	8	2297·943	3098·242	4095·043	5322·869	6817·148
7	5734608	29565		1695·437	2186·893	2783·005	3495·244	4324·181
		8570	7	3993·380	5285·135	6878·048	8818·113	11141·329
6	7190367	38135		2742·047	3489·486	4382·529	5421·897	6595·365
		10395	6	6735·427	8774·621	11260·577	1424·0010	17736·694
5	8956554	48530		4346·616	5459·020	6753·690	821·5399	9828·475
			5	11082·043	14233·641	18014·267	22455·409	27565·169
*	*	*	*	*	*	*	*	*

Here, as in the example of the last problem, to facilitate the subsequent summations, x is constant in the rows, and $x-y$ in the columns. The results, as they arise, have consequently to be written in the rows, commencing in each case in the first column, Diff. 0. There are in the example two columns headed x. The ages in the first of these columns are opposite the rows in the principal columns occupied by the successive terms of $d_x\lambda_y v^{x+1}$, and the ages in the second are opposite the rows occupied by the summations of the same function. The columns between those containing the values of x, are occupied, as shown, by the terms of two of the series whose formation has been exemplified.

The process of formation is as follows. To form the row corresponding to any given value of x, take the proper term of $d_x v^{x+1}$ for the in-factor, and multiply continuously by the adjoining term of λ_y and the differences of this series, commencing with $\Delta\lambda_{y-1}$. The regulator is at addition throughout, since the differences of λ_y are negative. A slip containing the differences should be placed on the machine in front of the operator; and to avoid error, each difference as it is used should be covered with a paper weight.*

Series for verification may be formed as follows :—

The function is

$$d_x\lambda_y v^{x+1}.$$

Hence if x vary,

$$\Delta_x(d_x\lambda_y v^{x+1})=\lambda_y.\Delta(d_x v^{x+1});$$

$$\therefore \quad d_{x+1}\lambda_y v^{x+2}=d_x\lambda_y v^{x+1}+\lambda_y\Delta(d_x v^{x+1});$$

or, changing x into $x-1$,

$$d_x\lambda_y v^{x+1}=d_{x-1}\lambda_y v^x+\lambda_y\Delta(d_{x-1}v^x).$$

The series thus indicated is one that, commencing in Column 0, $(x=y,)$ ascends diagonally. The formation, commencing with age 85, is as follows :—

* A better plan, perhaps, will be to insert the differences, as in the example, between the terms from which they arise.

x	$\Delta d_x v^{x+1}$	$\lambda_{85} d_x v^{x+1}$
85		4346616
6	1766187	3489486
7	1455759	2783005
8	1305032	2149673
9	968150	1679830
90	691471	1344259
1	601396	1052401
2	543116	7888280
3	413909	5879575
4	373082	4069008
5	334807	2444194
6	276214	1103718
7	177749	241107
	49682	000000

Taking λ_{85} for the in-factor, we have here for the initial term $\lambda_{85} \times d_{85} v^{86}$, $= 4346616$; and the succeeding terms are formed by continuous multiplication by the differences of the series $d_x v^{x+1}$, the regulator being set for subtraction. Verification is had by the term beyond 97 coming out equal to 0.

We have now to attend to the construction of the values of the function for the combinations in which x is equal to or less than y.

The function is,

$$d_x \lambda_y v^{y+1};$$

and if x vary we have,

$$\Delta_x (d_x \lambda_y v^{y+1}) = \lambda_y v^{y+1} . \Delta d_x .$$

Whence,

$$d_{x+1} \lambda_y v^{y+1} = d_x \lambda_y v^{y+1} + \lambda_y v^{y+1} . \Delta d_x ;$$

and for the formation in reverse order,

$$d_x \lambda_y v^{y+1} = d_{x+1} \lambda_y v^{y+1} - \lambda_y v^{y+1} . \Delta d_x .$$

And changing x into $x-1$,

$$d_{x-1} \lambda_y v^{y+1} = d_x \lambda_y v^{y+1} - \lambda_y v^{y+1} . \Delta d_{x-1} ,$$

which is the working formula.

The following is an example, showing the commencement of the process.

$x < y$

y	$\lambda_y v^{y+1}$	d_x	y	Diff. 0.	Diff. 1.	Diff. 2.	Diff. 3.	Diff. 4.
97	24841	9	97	·02236	·09936	·21363	·34529	·48440
6	164887	31 40		·65955	1·41803	2·29194	3·21531	4·18814
5	538783	46 86	6	·68191 4·63354	1·51739 7·48909	2·50557 10·50628	3·56060 13·68510	4·67254 17·72598
4	1233551	53 139	5	5·31545 17·14635	9·00648 24·05423	13·01185 31·33218	17·24570 40·58381	22·39852 50·32886
3	2308127	56 195	4	22·46180 45·00848	33·06071 58·62643	44·34403 75·93738	57·82951 94·17158	72·72738 114·2523
2	3814032	59 254	3	67·47028 96·87643	91·68714 125·4817	120·28141 155·6125	152·00109 188·7946	186·9797 234·5630
1	5849836	75 329	2	164·34671 192·4596	217·1688 238·6733	275·8939 289·5668	340·7957 359·7649	421·5427 452·1922
0	8527116	79 408	1	356·8063 347·9063	455·8421 422·0922	565·4607 524·4176	700·5606 659·1460	873·7349 802·4015
89	1194017	87 495	0	704·7126 591·0384	877·9343 734·3205	1089·8783 922·9751	1359·7066 1123·570	1676·1364 1358·791
8	1629580	120 615	89	1295·7510 1002·191	1612·2548 1259·665	2012·8534 1533·434	2483·277 1854·461	3034·927 2193·414
7	2193321	158 773	8	2297·942 1695·437	2871·920 2063·915	3546·287 2495·999	4337·738 2952·209	5228·341 3388·680
6	2913971	168 941	7	3993·379 2742·047	4935·835 3316·099	6042·286 3922·204	7289·947 4502·085	8617·021 5009·116
5	3819522	197 1138	6	6735·426 4346·615	8251·934 5141·076	9964·490 5901·161	11792·032 6565·757	13626·137 7192·159
			5	11082·041	13393·010	15865·651	18357·789	20818·296
*	*	*	*	*	*	*	*	*

The formation here is in such entire accordance with that of $d_x\lambda_y v^{x+1}$, already given, that explanation of it would seem to be superfluous. Comparison of the two examples will show that they differ only by the interchange of x and y, $d_x v^{x+1}$ and $\lambda_y v^{y+1}$, and λ_y and d_x. The process in both is the same. The leading term of each row is the product of the values of the two functions in line with it; and the succeeding terms are produced by the continuous employment as multipliers of the series of differences in the column adjoining that headed Diff. 0.

The formation of the verification series here is also in accordance with that of the series for the verification of $d_x\lambda_y v^{x+1}$.

y	$\Delta\lambda_y v^{y+1}$	$d_{85}\lambda_y v^{y+1}$
85		4346615
6	905551	3316099
7	720650	2495999
8	563741	1854461
9	435563	1358791
90	341305	9703857
1	267728	6657112
2	203581	4340369
3	150590	2626649
4	107458	1403780
5	69477	613136
6	37389	187642
7	14005	28269
	2484	00000

The formation of the functions $d_x\lambda_y v^{x+1}$ and $d_x\lambda_y v^{y+1}$ having been completed, there remain the summations of the terms to be performed. This is rendered a very simple matter in consequence of the arrangement adopted, which is that suggested by General Hannyngton in the case of the construction of $D_{x \cdot y}$. To ensure accuracy the columns should first be added in groups of ten or fifteen terms; and the sums being carried forward, will serve as checks in the final summation.

It is the results of these final summations that are tabulated, each of them having reference to the entire after-lifetime of the combination opposite which it is placed.

To suit the form of the table, it is requisite now to make a change in the nomenclature that up to this point has been employed. The symbols x and y have been used hitherto to denote the ages of the life assured and the life assured against, respectively. This appropriation of the symbols was convenient so long as we were concerned with merely the construction of the terms; but it is otherwise when we come to consider the arrangement which will most facilitate the use of the table. The only tables of this kind that have been published are the valuable and extensive tables of Mr. David Chisholm, founded on the Carlisle Table of Mortality; and we cannot do better than adopt the arrangement and notation of which Mr. Chisholm has set the example in his work. In the sequel, therefore, when the ages are other than equal, x will stand for the older of the two, and y for the younger; and the two sets of results will be arranged in parallel columns, designated respectively by the symbols $M_{\overline{x.y}}^{1}$ and $M_{\overline{x.y}}^{\;\;1}$, the analogy of which to those which denote the values of Survivorship

Assurances on (x) and (y) is apparent. We may also, if we please, form a column $M_{x \cdot y}$, by adding together the corresponding terms in the two preceding columns.

The following is a specimen of the final arrangement.

Diff. 2.

x	y	$M_{\overset{1}{x}.y}$	$M_{x.\overset{1}{y}}$	$M_{x.y}$
*	*	*	*	*
85	83	18014·27	15865·65	33879·92
86	84	11260·58	9964·49	21225·07
87	85	6878·05	6042·29	12920·34
88	86	4095·04	3546·29	7641·33
89	87	2405·82	2012·85	4418·67
90	88	1382·45	1089·88	2472·33
91	89	755·751	565·461	1321·212
92	90	385·470	275·894	661·364
93	91	181·314	120·281	301·595
94	92	73·790	44·344	118·134
95	93	23·818	13·012	36·830
96	94	5·108	2·506	7·614
97	95	·457	·214	·671

In vol. v of the *Journal*, pp. 107 to 118, will be found a prototype of the operation that forms the subject of the present problem.

It is in place to mention that the examples here given, like those in illustration of Problem VII. are pointed to correspond to a radix of 10,000.

I have now completed the task I proposed to myself when commencing the preparation of the present series of papers. I have, I think, shown that the Arithmometer possesses a singular adaptation to the construction of the tables required for Actuarial use. The examples I have given by no means exhaust the capabilities of the instrument in this department; but the necessity for here going further into the subject is precluded by the publication, since my papers were commenced, of Mr. Ralph P. Hardy's remarkable work, entitled *Valuation Tables, based upon the Institute of Actuaries' Mortality Experience* (H^M) *Table*. Mr. Hardy's work contains complete tables, at four rates of interest, of most of the functions which I have chosen for examples, and of others besides, nearly all of them having been calculated by the aid of the Arithmometer. It thus forms an extensive repertory of examples for the use of such as desire to master the working of the machine, and to elicit its further capabilities. Had the work referred to been published, or had I known that it was in contemplation, before I commenced the preparation of my papers, they would, if they had appeared at all, have assumed a different form from that which they now present. Mr. Hardy's work is one of unquestionable utility, and will form a lasting monument of the author's skill and enterprise.

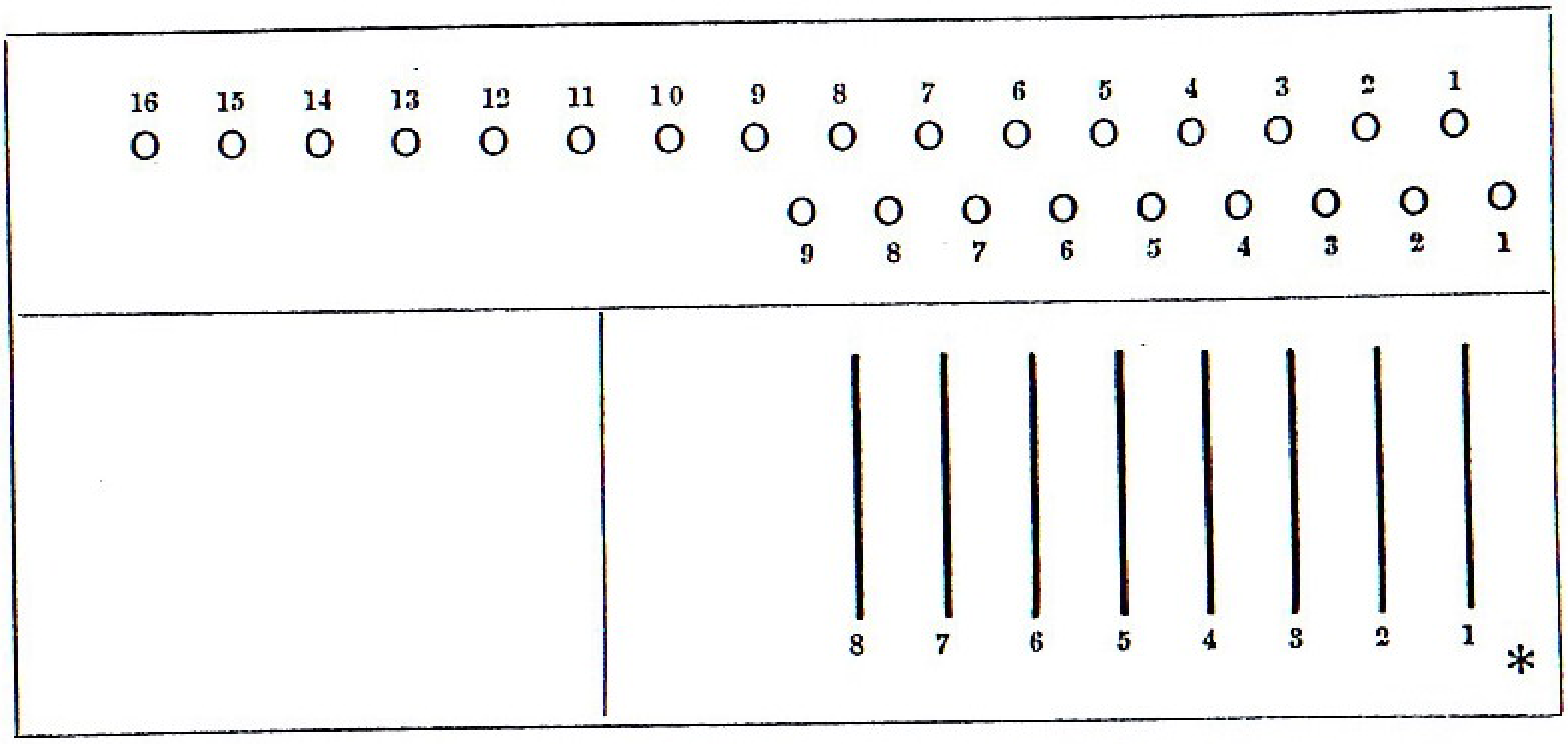
It will be sufficiently apparent, from what I have said, that I quite concur in the laudatory terms in which Mr. Hardy, in his preface, speaks of the Arithmometer. I also agree with him in what he says as to the existence of room for improvement in the strength and temper of certain of the materials employed. This refers chiefly to the springs, which form the weak point of the machine. Any improvement in form or material given to these, which should remove or diminish the tendency to give way that they sometimes exhibit, would be hailed as a boon by all users of the machine.

I hope it will not be deemed presumptuous if now, ere I close, I venture respectfully to invite the attention of the makers of the machine to two points, in regard to which I consider some modification is desirable.

The first has reference to the manner in which the result (when the process in use is multiplication) is presented. The figures composing it are seen at the bottom of a series of holes, that which I have designated S_1, being the upper series on the slide. These holes are of the form of truncated hemispherical cavities, a quarter of an inch deep (such being the thickness of the slide), and five-eighths of an inch in diameter; and a consequence of this form is, that in working, with the paper at the right hand, and especially if the numbers being dealt with are *long* ones, more movement of the body, and more stooping over the machine than is at all agreeable, become necessary, to avoid mistake in reading off the figures. A position and direction of the light, also, which, probably, cannot always be conveniently commanded, are required for a like reason.

The remedy for these inconveniences—for such I believe they will be admitted to be by all who have had occasion to use the machine extensively—would be, of course, a reduction in the depth of the cavities at the bottom of which the figures appear. I do not presume to suggest the manner in which this might be accomplished; I will only say that I see no insurmountable mechanical difficulty in the way.

The second point to which I refer, is the desirableness of having the openings through which the figures appear individually numbered. To render my meaning plain, I give here a plan of the machine, with the figures that I propose inserted.



* *Note.*—The asterisk indicates the position of the handle.

I frequently feel the want of this provision in setting factors on the machine, having, in its absence, to count the holes; and also in performing multiplication from left to right.† I believe that those practically acquainted with the machine will agree with me in thinking that the additions I have suggested would facilitate its use.

† I find this the more convenient order, and I always practise it when it is *safe* to do so—that is, when the number on S_1 does not extend beyond the limit of the carrying power of the machine; and also even when it does so extend, provided that the leading figures change but slowly.

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